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Applying the Poincaré Recurrence Theorem to Billiards

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Abstract

The Poincaré recurrence theorem is one of the first and most fundamental theorems of ergodic theory. When applied to a dynamical system satisfying the theorem's hypothesis, it roughly states that the system will, within a finite amount of time, return to a state arbitrarily close to its initial state. This result is intriguing and controversial, providing a contradiction with the Second Law of Thermodynamics known as the recurrence paradox. Here, we treat a set of pool balls on a billiard table as a dynamical system that satisfies the hypotheses of the Poincaré recurrence theorem. We prove that time is a volume-preserving transformation from the state space onto itself. After showing that the hypotheses for the recurrence theorem are met, we discuss the theorem's implications, including possible resolutions of the paradox that arises.

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1 Introduction

The Poincaré recurrence theorem is one of the most basic theorems of ergodic theory and provides very important consequences for dynamical systems. When applied to a dynamical system satisfying the theorem's premises, it roughly states that the system will, within a finite amount of time, return to a state arbitrarily close to its initial state. For Hamiltonian systems, the premises of the Poincaré recurrence theorem are often hastily satisfied by citing Liouville's theorem [2]. However, they may also be satisfied directly, using only undergraduate-level mathematics. We will show that by applying correct assumptions and restrictions, it is possible to directly satisfy the premises of the Poincaré recurrence theorem with a set of pool balls on a billiard table. This setting provides an interesting platform for discussion of the theorem's result, including possible resolutions of the paradox that arises.

In order to formally state and prove the Poincaré recurrence theorem, we must give mathematical meaning to the concept of recurrence.

Definition 1. *Suppose a dynamical system has state space S, and the passage of time is defined by a discrete transformation $T\colon S \to S$. Let $A \in S$ be any arbitrary subset of S. Then, a point $x \in A$ is said to* ***recur*** *to A with respect to T if there exists a natural number $n \in \mathbb{N}$ such that $T^n(x) \in A$.*

Here, exponentiation of the transformation T represents composition of T with itself. The natural number n represents the number of discrete time intervals that must pass before recurrence of x is observed. Using this definition of recurrence, we can give a formal statement and proof of the theorem.

Poincare Recurrence Theorem. *If S is a bounded space with measure μ and $T\colon S \to S$ is a measure-preserving transformation, then for any set with positive measure $B \subseteq S$, the subset $A \subseteq B$ of points that never recur to B has measure zero [4].*

Proof. Suppose S is a bounded space with measure μ such that S has finite, positive measure. Additionally suppose $T\colon S \to S$ is a measure-preserving transformation on S and $B \subseteq S$ is an arbitrary subset with positive measure. Define $A = \{x \in B \mid \forall k \in \mathbb{N}, T^k(x) \notin B\}$ as the subset of points in B that never recur to B. We wish to prove $\mu(A) = 0$. To do this, consider the preimages of A. We proceed by contradiction to show that these preimages are mutually disjoint. Suppose $x \in T^{-m}(A) \cap T^{-n}(A)$ for some natural numbers m and n, where $m > n$. But then the point $T^n(x)$ recurs to $A \subseteq B$, since $T^n(x) \in A$ and $T^{m-n}(T^n(x)) = T^{m-n+n}(x) = T^m(x) \in A$. Since A is the set of points that never recur to B, no such x exists. Thus, preimages of A are mutually disjoint.

Next, note that $\bigcup_{n=1}^{\infty} T^{-n}(A) \subset S$, implies $\mu\left(\bigcup_{n=1}^{\infty} T^{-n}(A)\right) \leq \mu(S)$. Furthermore, the measure of a union of disjoint sets is equal to the sum of the measures of the sets and T preserves measure; thus

$$\mu\left(\bigcup_{n=1}^{\infty} T^{-n}(A)\right) = \sum_{n=1}^{\infty} \mu(T^{-n}(A)) = \sum_{n=1}^{\infty} \mu(A) \leq \mu(S)$$

Finally, note that this is an infinite series with constant terms whose sum is bounded by $\mu(S) < \infty$. This is only possible if $\mu(A) = 0$.

□

The billiard system that we construct serves as the example through which this theorem's result is discussed. To satisfy the theorem's premises, we must assume that the pool table is bounded and frictionless, energy is conserved, and all energy is kinetic. Additionally, we simplify the calculations by assuming all ball collisions occur instantaneously.

2 Constructing a Bounded State Space

To apply the recurrence theorem, we must first have a bounded, measurable space. Recall that we wish to use the theorem to show that the billiard system will eventually return to a state close to its initial state. This implies that the system's state space will serve as the bounded, measurable space S.

Definition 2. *The state space of a system is the set of all allowable states of that system.*

Suppose a dynamical system consists of n billiard balls of equal mass m existing on a bounded billiard table. To construct the state space for this system, we first apply a two-dimensional spacial Euclidean coordinate system to the table. Since the table is bounded, choose one corner as the origin and assume the spacial coordinates vary from $(0, 0)$ to (a, b), as shown in Figure 1. The position of each ball in the system can be represented by

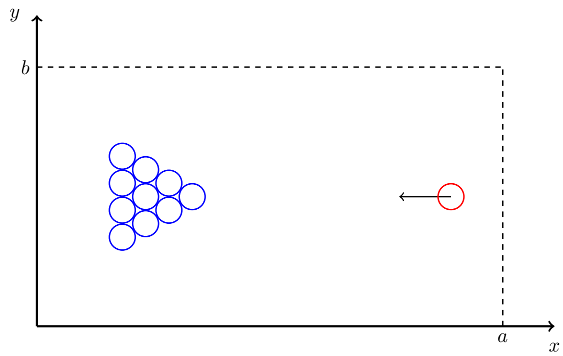

Figure 1: Bounded billiard table in a possible initial state

two coordinates; therefore, denote x_i and y_i as the position coordinates of the i^{th} ball. However, at a given instant in time, the positions of the balls in the system are not sufficient to describe the entire state of the system including all future and past behavior; we need velocities as well. Thus, let v_{xi} and v_{yi} denote the velocity coordinates of the i^{th} ball.

The collective knowledge of the positions and velocities of every ball at a particular instant in time constitutes sufficient information to describe the entire state of the system at that instant. Because the system is deterministic, this includes knowledge of the unique future and past behavior of the system. Thus, we require 4 real numbers to fully describe each ball, or $4n$ real numbers to describe the entire system. This indicates that the state space of the system is given by $S \subseteq \mathbb{R}^{4n}$.

The position coordinates of each ball in the state space are bounded below by 0 and above by $a > 0$ (for x-coordinates) or $b > 0$ (for y-coordinates), due to the geometric construction of the billiard table; a ball with position coordinates outside of these bounds would be outside of the physical table boundaries, which is disallowed. However, for the entire state space to be bounded, we must also show that the velocity coordinates are bounded.

Although the velocity coordinates are not intrinsically bounded, they attain bounds when the system is given an initial state as a byproduct of conservation of energy in the system. To show this, suppose the system is given an initial state and allowed to propagate in time. In this respect, let $\vec{\mathbf{x}} \in S$ be the initial state of the system. As time passes, the positions and velocities of the balls become functions of time. Next, note that the balls travel linearly during intervals without collisions; mathematically, the derivatives of the balls' position functions with respect to time are the balls' velocity functions, and the derivatives of the balls' velocity functions with respect to time are zero. This describes the following first order piecewise continuous system of ordinary differential equations:

$$\begin{cases} \frac{dx_i(t)}{dt} & = & v_{xi}(t) & \text{for } i = 1, 2, 3, \ldots, n \\ \frac{dy_i(t)}{dt} & = & v_{yi}(t) & \text{for } i = 1, 2, 3, \ldots, n \\ \frac{dv_{xi}(t)}{dt} & = & 0 & \text{for } i = 1, 2, 3, \ldots, n \\ \frac{dv_{yi}(t)}{dt} & = & 0 & \text{for } i = 1, 2, 3, \ldots, n \end{cases} \tag{1}$$

These equations are piecewise continuous since they only apply during intervals without collisions, and collisions are assumed to be instantaneous. If the solution to this system of equations is given by

$$\begin{aligned} \vec{\mathbf{r}}(t) = (&x_1(t), y_1(t), x_2(t), y_2(t), \ldots, x_n(t), y_n(t), \\ &v_{x1}(t), v_{y1}(t), v_{x2}(t), v_{y2}(t), \ldots, v_{xn}(t), v_{yn}(t)) \end{aligned} \tag{2}$$

then the initial system state $\vec{\mathbf{x}} \in S$ becomes the initial condition $\vec{\mathbf{r}}(0) = \vec{\mathbf{x}}$ to an initial-value problem. Thus, by the existence and uniqueness theorem for first-order ordinary differential equations, the solution $\vec{\mathbf{r}}(t)$ is guaranteed to exist and be unique [1].

Next, consider the total energy of the system. All energy is kinetic by assumption, which implies that the total energy of the system at any time t may be expressed by the sum of the kinetic energies of the balls. Mathematically,

$$E(t) = \sum_{i=1}^{n} \frac{1}{2} m v_i^2 = \frac{m}{2} \sum_{i=1}^{n} \left[\sqrt{v_{xi}^2(t) + v_{yi}^2(t)} \right]^2 = \frac{m}{2} \sum_{i=1}^{n} \left[v_{xi}^2(t) + v_{yi}^2(t) \right] \tag{3}$$

where v_i represents the speed of the i^{th} ball. Since we insist that energy is conserved, the total energy in the system is constant at all points on the curve $\vec{\mathbf{r}}(t)$. Thus, if the total system energy at the initial state is denoted by E, then we may rearrange equation (3) to obtain

$$\sum_{i=1}^{n} \left[v_{xi}(t)^2 + v_{yi}(t)^2 \right] = \frac{2E}{m} \tag{4}$$

which holds for every point on $\vec{\mathbf{r}}(t)$.

Equation (4) describes a level surface in $\mathbb{R}^{4n}$ [5]. Since the system always satisfies this equation, every point in the state space must lie on this level surface; in other words, S is a subset of this level surface. This implies that every velocity coordinate of S is bounded above by $\sqrt{\frac{2E}{m}}$ and below by $-\sqrt{\frac{2E}{m}}$.

At this point, it is clear that the state space S is bounded. Indeed, the position coordinates of S are intrinsically bounded by the construction of the table; additionally, applying an initial state to the system imposes a bound on the velocity coordinates. However, in order to apply the theorem, S must also be measurable. Since S is a subset of $\mathbb{R}^{4n}$, classical Euclidean volume in $\mathbb{R}^{4n}$ serves as the measure μ on S. Note that S clearly has positive, finite volume.

3 Time as a Volume-Preserving Transformation

In addition to a bounded, measurable space, the Poincaré recurrence theorem requires a measure-preserving transformation T from the space onto itself. So far, time has served as the independent variable for a system of ordinary differential equations. We now turn to characterizing time in a more general form as a collection of volume-preserving transformations that satisfy the hypotheses of the theorem.

Thus, define the passage of time by a family of transformations from the state space onto itself. Formally, for all real numbers t, define $T_t \colon S \to S$ such that if $\vec{\mathbf{s}} \in S$ is any system state, then $T_t(\vec{\mathbf{s}}) \in S$ represents the system state t seconds later. Note that the Poincaré recurrence theorem only requires a single measure-preserving transformation T; we show that this criterion is met by every T_t in this family of functions. Thus, to apply the theorem, we simply choose a discrete time step $\alpha > 0$, and use T_α as the transformation T.

Now, T_t is a volume-preserving transformation if and only if for any measurable region $R \subseteq S$, the volume of R is the same as the volume of the image of R, namely the region $T_t(R)$. However, for any continuous transformation, we may express the volume of a region as it is transformed by

$$\int_{R} dA = \int_{T_t(R)} |J[T_t]| \, dA \tag{5}$$

where $J[T_t]$ represents the Jacobian of T_t [5]. In equation (5), $|J[T_t]|$ represents the factor by which the region must be multiplied in order for the equality to hold. If $|J[T_t]| = 1$, then the equation reduces to

$$\int_{R} dA = \int_{T_t(R)} dA \tag{6}$$

which expresses that the volume of R is equal to the volume of $T_t(R)$. This implies that T_t is volume-preserving if and only if $J[T_t] = \pm 1$. Thus, we must calculate the Jacobian of T_t. We consider two cases: time intervals where no collisions occur and the instant when a collision occurs. In both cases, T_t is linear; this implies that the Jacobian of T_t is simply the determinant of the matrix representation of T_t in each case.

3.1 Preserving Volume During Intervals Without Collisions

We first consider time intervals where there are no ball collisions. Given an initial point and assuming that no ball collisions occur during an interval of time t, elementary physics dictates that the velocities of the balls will remain constant while the positions will increase by vt. Thus, T_t has the form

$$T_t\left(\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_n \\ y_n \\ v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \\ \vdots \\ v_{xn} \\ v_{yn} \end{bmatrix}\right) = \begin{bmatrix} x_1 + tv_{x1} \\ y_1 + tv_{y1} \\ x_2 + tv_{x2} \\ y_2 + tv_{y2} \\ \vdots \\ x_n + tv_{xn} \\ y_n + tv_{yn} \\ v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \\ \vdots \\ v_{xn} \\ v_{yn} \end{bmatrix} \tag{7}$$

Clearly, the matrix representation of T_t that elicits this behavior is given by

$$T_t\left(\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_n \\ y_n \\ v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \\ \vdots \\ v_{xn} \\ v_{yn} \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 & 0 & t & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 & t & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & 0 & t & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & 0 & 0 & t & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & 0 & 0 & 0 & \cdots & t & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & t \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ \vdots \\ x_n \\ y_n \\ v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \\ \vdots \\ v_{xn} \\ v_{yn} \end{bmatrix} \tag{8}$$

Since this matrix is upper-triangular, its determinant is the product of its diagonal entries.

$$J[T_t] = det(T_t) = \prod_{i=1}^{n} T_{t,ii} = \prod_{i=1}^{n} 1 = 1 \tag{9}$$

Thus, the determinant of T_t is equal to 1 for intervals where no ball collisions occur, which implies that the transformation preserves volume during such intervals.

3.2 Preserving Volume At the Instant of a Collision

It is slightly more complicated to show that T_t preserves volume during intervals when a collision occurs. However, the proof may be simplified by noticing that an interval of time during which a collision occurs may be decomposed into the subinterval before the collision, the instantaneous collision itself, and the subinterval after the collision. We have already shown that volume is preserved during the subintervals, since no collisions occur during them. Therefore, we must only show that volume is preserved during the *instant* the collision occurs. Since we assume collisions occur instantaneously, it is reasonable to assume that no more than two balls collide at any given instant; indeed, the probability that two such collisions will occur simultaneously is zero. Thus, T_0 must map the instant before a collision to the instant immediately afterwards. Finally, it is important to note that since these two points occur concurrently, we expect the transformation to affect the velocities of the two balls but not the positions.

In order to simplify the calculation of the balls' final velocities, we first employ a rotation transformation matrix to rotate the initial velocity vectors into a $u - v$ coordinate system where the collision is reduced to a one-dimensional problem (see Figure 2). The angle of rotation is chosen so that the force of the balls colliding, which affects the final velocities of the balls, is entirely in the u-direction of the new coordinate system. Thus, elementary physics dictates that the two balls simply switch their u-component velocities, while retaining the same v-component velocities as before [3]. After applying this velocity switch, we rotate the vectors back to the original $x - y$ coordinate system. For $n = 2$, we use the rotational matrix $\boldsymbol{R_\theta}$ to rotate the velocities into the desired coordinate system, the row-switching matrix $\boldsymbol{W}$ to swap the u-components of the velocities, and finally $\boldsymbol{R_{-\theta}}$ to rotate the velocities back into their original coordinate system.

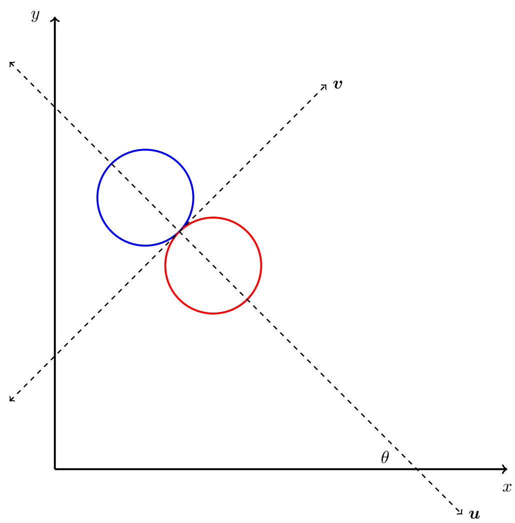

Figure 2: Coordinate transformation to simplify calculations during a collision

$$\boldsymbol{R_\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & \sin\theta & \cos\theta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & 0 & 0 & 0 & 0 & \sin\theta & \cos\theta \end{bmatrix} \tag{10}$$

$$\boldsymbol{W} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Assuming a collision occurs between balls 1 and 2 at the point specified, the final transformation is given by

$$T_0\left(\begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \end{bmatrix}\right) = \boldsymbol{R_{-\theta}} \times \boldsymbol{W} \times \boldsymbol{R_{\theta}} \times \begin{bmatrix} x_1 \\ y_1 \\ x_2 \\ y_2 \\ v_{x1} \\ v_{y1} \\ v_{x2} \\ v_{y2} \end{bmatrix} \tag{11}$$

By the chain rule, the Jacobian of T_0 is simply the product of the determinants of these matrices. However, since rotations have determinant 1 and row switches have determinant -1, the Jacobian of T_0 is

$$J[T_0] = det(T_0) = det(\boldsymbol{R_{-\theta}}) \times det(\boldsymbol{W}) \times det(\boldsymbol{R_{\theta}}) = (1)(-1)(1) = -1 \tag{12}$$

Finally, note that we have so far focused only on the two colliding balls and ignored the presence of the other balls in the system. However, since T_0 is an *instantaneous* transformation, we expect the coordinates of the other balls in the system to remain fixed. This indicates that the full matrix representation of T_0 during a collision mirrors the identity matrix for the omitted entries, with ones on the diagonals and zeros elsewhere. This implies that the omitted entries will not affect the overall determinant of the matrix, which will still be -1. Thus, T_t is a volume-preserving transformation during intervals with collisions as well.

4 Discussion

Now that we have proved the billiards system satisfies the hypotheses of the Poincaré recurrence theorem, we have a setting by which to discuss the theorem's result.

Suppose one views this billiard table and observes an initial configuration like the state pictured in Figure 1. Consider the set $B \in S$ to be an ε-ball centered around the initial configuration observed. If $\varepsilon > 0$ is small enough, then every point in B represents a state that is imperceptibly close to the initial state - so close that one could not distinguish between them with the naked eye. However, since B is a ball, it clearly has positive volume. Next, he or she fixes a discrete time interval, say one hour, and periodically checks the table once per interval. The Poincaré recurrence theorem states that the set of points in B that will never return to B no matter how many time intervals are observed has measure zero. This means that with probability one, the system's initial configuration is not in this subset, and eventually will return to B. Therefore, after a finite number of intervals, one will check the table to find that it has returned to a state in B; that is, it has returned to a state imperceptibly close to the initial state. If the initial state is that the balls are "racked" together and the cue ball is about to "break" them, then the balls will, within a finite amount of time, spontaneously appear to re-rack and re-break themselves. This is called "almost periodic" behavior.

The Poincaré recurrence theorem is controversial when considered within the context of the Second Law of Thermodynamics.

Second Law of Thermodynamics. *The entropy of an isolated system (or group of systems) never decreases. The entropy either increases, until the system reaches equilibrium, or, if the system began in equilibrium, stays the same [3].*

According to the Second Law of Thermodynamics, the measure of disorder of a system will never decrease - it will either increase or stay the same. However, this directly contradicts the result of the recurrence theorem, which claims that an isolated system like the billiard table in a high state of disorder, such as the state directly after the break, might once again reorder itself as it spontaneously re-racks into a state of decreased entropy. This is known as the recurrence paradox and is most commonly reconciled by a claim that the amount of time that one must wait before the billiard system returns to its initial state is orders of magnitude larger than the expected life of the universe.

It is notable to mention that if the set B has a smaller measure, one can generally expect to wait more time steps for the system to return than if B has a larger measure. This intuitive result comes from the observation that the largest possible measure of B would occur if $B = S$, in which case recurrence would be guaranteed after a single time step.

Similarly, the length of the time quantum α can also be correlated with the amount of time one can generally expect to wait before the system recurs. For instance, if one compares the total amount of time required to observe recurrence using a yearly time quantum with the amount of time required using a time step of one second, it would generally take longer to observe recurrence with the yearly quantum.

Finally, although the recurrence theorem guarantees recurrence *with probability one*, it does not guarantee that the system recurs for every single point in B. In fact, there may be infinitely many points that do not return to B. However, the theorem does assert that the set of points in B that do not return to B has measure zero – even though B has positive measure. Thus, although it is possible that the billiard table's initial configuration is one that never recurs, it is a zero probability event; consequently, the probability of recurrence is one.

5 Conclusion

After introducing the Poincaré recurrence theorem and briefly discussing its purpose, we demonstrated that a set of billiard balls on a table can be modeled as a dynamical system. Next, we employed this system as an example to exhibit the theorem's results. During this process, we applied a number of simplifying assumptions including the presumption that collisions are instantaneous. We hypothesize that this restriction may be relaxed without affecting the result. In addition, the Poincaré recurrence theorem may be applied to a myriad of similar systems in addition to billiards. For instance, the theorem may be applied to the random motion of gas molecules in a chamber, to prove that the particles will spontaneously collect on one half of the chamber. However, although this system is also Hamiltonian, it is more complicated than billiards because it involves 3-dimensional motion, while billiard balls are restricted to 2-dimensional motion. The Poincaré recurrence theorem is known for its surprising result; it is an interesting theorem that provides a glimpse into ergodic theory.

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